Analysis 1

9 April 2024

Slope

What is the slope of $y = 1.5x + 9.3$?

- Answer: 1.5, or 3/2

What is the *meaning* of this fact? (What is the definition of “slope”?)

- Answer: for a point on $y = 5x + 9$,
  - If you increase x by 1, you increase y by 1.5.
  - If you increase x by 2, you increase y by 3.
  - If you increase x by 26, you increase y by 39.
  - If you increase x by 0.01, you increase y by 0.015.

Last time

Idea: $f'(5)$ is a *number* that is the *slope* of the tangent line to $y = f(x)$ through the point $(5, f(5))$.

Idea: $f'(x)$ is a *function* that gives the derivative for various x-values.

- Also written f' $\quad Df$ $\quad \frac{\mathrm{d}}{\mathrm{d}x}f$ $\quad \frac{\mathrm{d}f}{\mathrm{d}x}$ $\quad \frac{\mathrm{d}y}{\mathrm{d}x}$

Calculations: We can find formulas for $f'(x)$ using

- $(x^p)' = px^{p-1}$ if p is constant
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$

and algebra (e.g., $\sqrt{x} = x^{1/2}$) and "derivative rules":

- For any constant c and function f, $\quad (cf)' = c \cdot f'$.
- For any functions f and g, $\quad (f + g)' = f' + g'$
- For any functions f and g, $\quad (f \cdot g)' = f'g + fg'$

Idea: $f'(5)$ is a *number* that is the *slope* of the tangent line to $y = f(x)$ through the point $(5, f(5))$.

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Questions:

What tasks can we answer using the derivative at a specific point?

What tasks can we answer using the derivative function?

Task 1: Give the slope of the tangent line to $y = 3 + \sqrt{2\pi x}\cos(x)$ at $x = \frac{\pi}{2}$.

By definition, this is $f'(\frac{\pi}{2})$ for the fn. $f(x) = 3+\sqrt{2\pi x}\cos(x)$.

Answer: $-\pi$

Task 2: Give an equation for the tangent line to $y = 3 + \sqrt{2\pi x}\cos(x)$ at $x = \frac{\pi}{2}$.

We will also need to use the number $f(\frac{\pi}{2}) = 3$.

This is the same as "give an eqn. for the line through the point $(\pi/2, 3)$ with slope $-\pi$".

Answer: $y = 3 + (-\pi)(x - \frac{\pi}{2})$, or $y = -\pi x + \frac{\pi^2}{2} + 3$

For more about line equations and slope (not using derivatives at all), see List 0: Review.

8. If a point on the line
$$y = -\tfrac{1}{3}(x-6)+8$$
has an x-value of 15, what is its y-value?

9. If a point on the line
$$y = -\tfrac{1}{3}(x-6)+8$$
has an x-value of 6, what is its y-value?

10. Graph each of the following:
    (a) $y = 3(x-1)+2$
    (b) $y = 3x-1$
    (c) $y-2 = 3(x-1)$
    (d) $y+1 = 3x$
    (e) $3x-y=1$
    (f) $x = (y+1)/3$

11. Give an example of a point that is on the line
$$y-17 = 38(x-12).$$

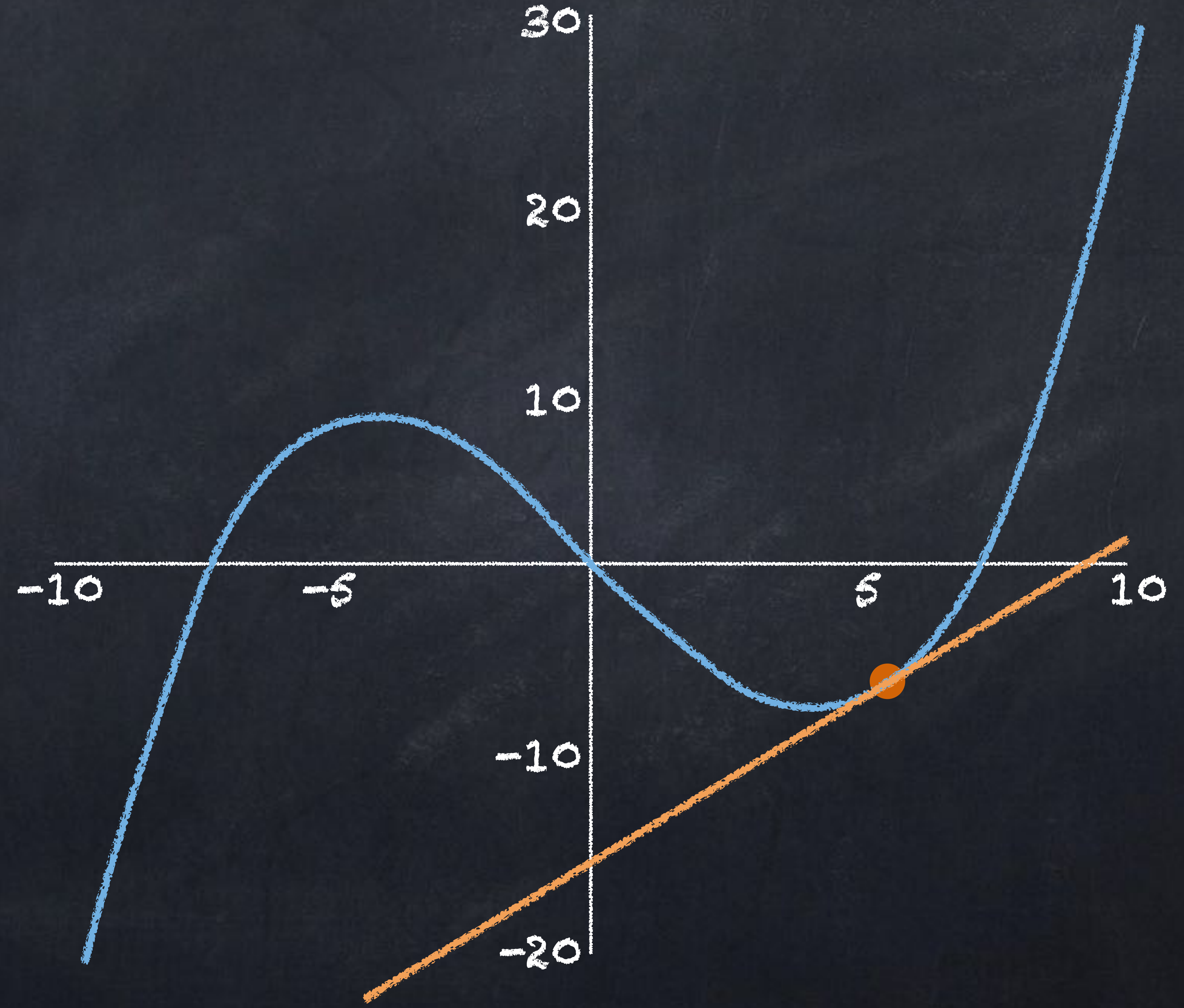
12. Describe the shape of $y = 7$ in words. Describe $x = -2$ in words.

13. Give an equation for the line through the point $(-6,5)$ with slope 2.

14. Give an equation for each of the following:
    (a) the line through $(1,3)$ with slope 5.
    (b) the line through $(0,-9)$ with slope $\frac{2}{5}$.
    (c) the line through $(-4.2,\ 6.1)$ with slope 8.88.
    (d) the line through $(5,1)$ with slope -3.

Calculation vs. ideas

Task: Find the tangent line to $y = \frac{1}{16}x^3 - 3x$ at $(6, \frac{-9}{2})$.



Calculation vs. ideas

Task: Find the tangent line to $y = \frac{1}{16}x^3 - 3x$ at $(6, \frac{-9}{2})$.

$$f(x) = \frac{1}{16}x^3 - 3x$$

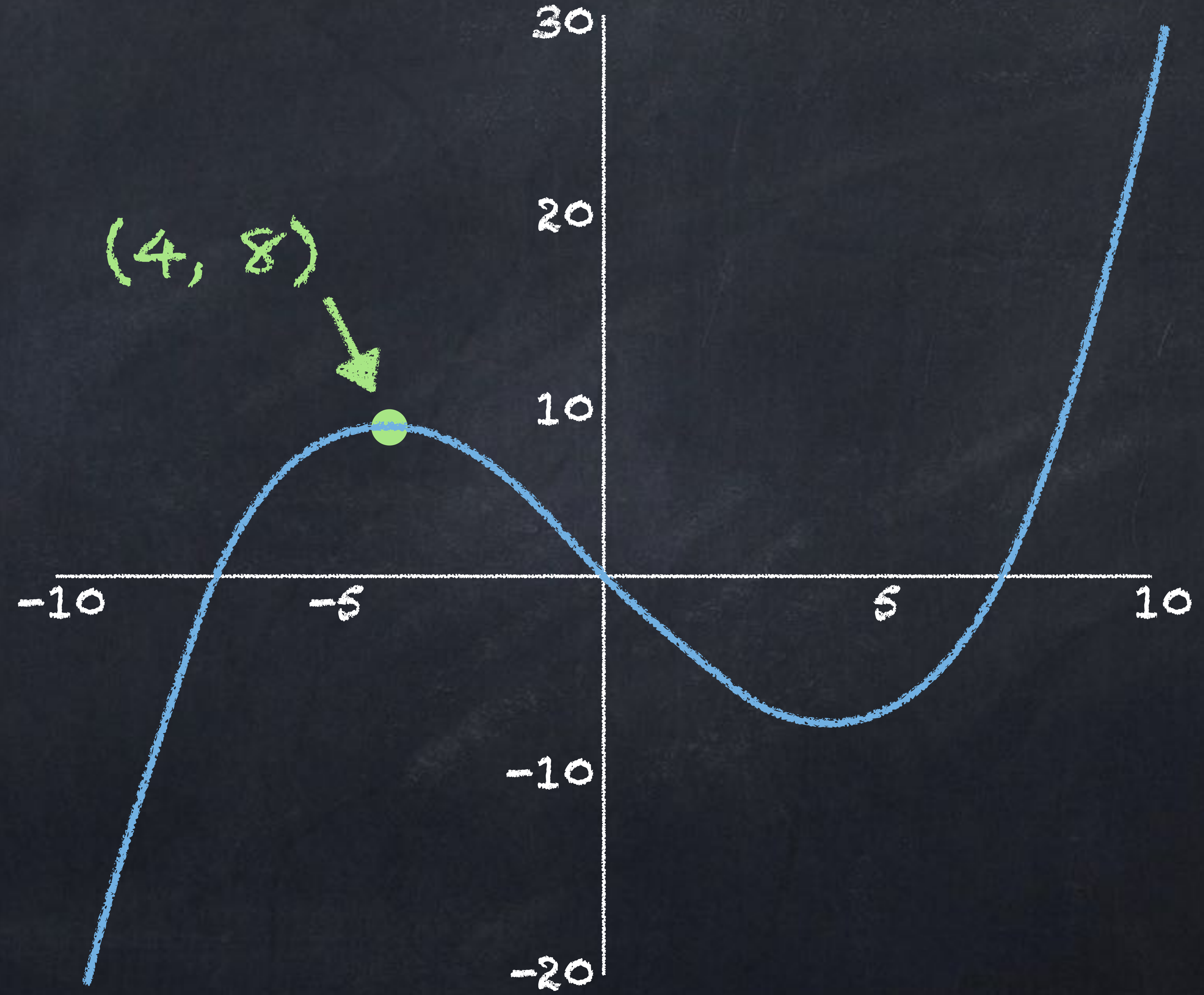
$$f'(x) = \frac{3}{16}x^2 - 3$$

$$f'(6) = \frac{3 \times 36}{16} - 3 = \frac{15}{4}$$

$$y = \frac{-9}{2} + \frac{15}{4}(x - 6)$$

Calculation vs. ideas

Task: Find the "local maximum" of $\frac{1}{16}x^3 - 3x$.



Calculation vs. ideas

Task: Find the "local maximum" of $\frac{1}{16}x^3 - 3x$.

$$f(x) = \frac{1}{16}x^3 - 3x$$

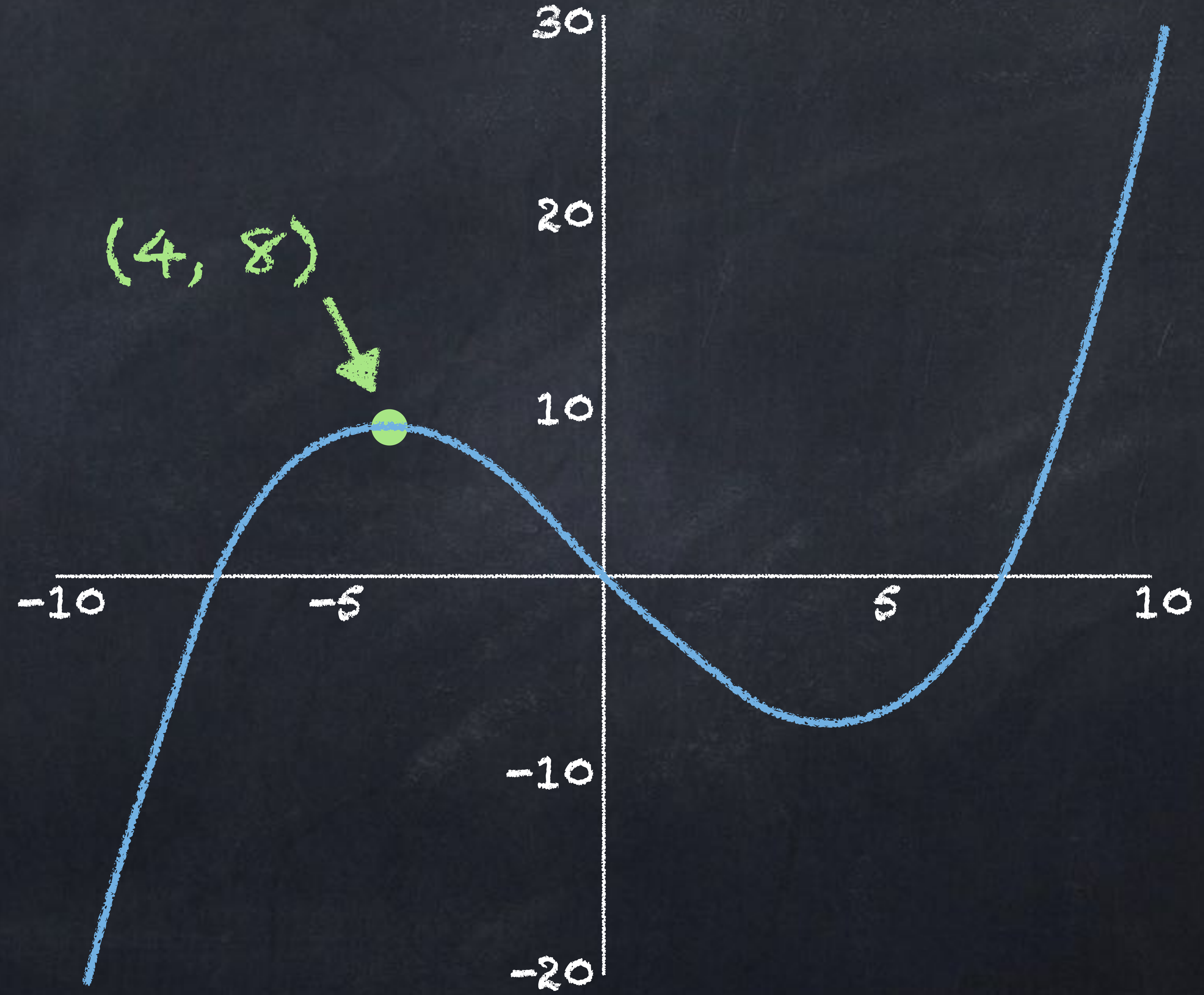
$$f'(x) = \frac{3}{16}x^2 - 3$$

...?...?...?...

(we will learn this later)

$$x = -4$$

$$f(-4) = \frac{1}{16}(-4)^3 + 12 = 8$$



Increasing and decreasing

a. What is the value of

$$f(x) = 5\sqrt{x}$$

when $x = 4$?

b. If x increases from 4 (becoming slightly more than 4), does $f(x)$ increase or decrease?

c. ~~If x decreases from 4, does $f(x)$ increase or decrease?~~

d. What is the value of $g(x) = \dfrac{6}{x}$ when $x = 2$?

e. If x increases from 2, does $g(x)$ increase or decrease?

f. ~~If x decreases from 2, does $g(x)$ increase or decrease?~~

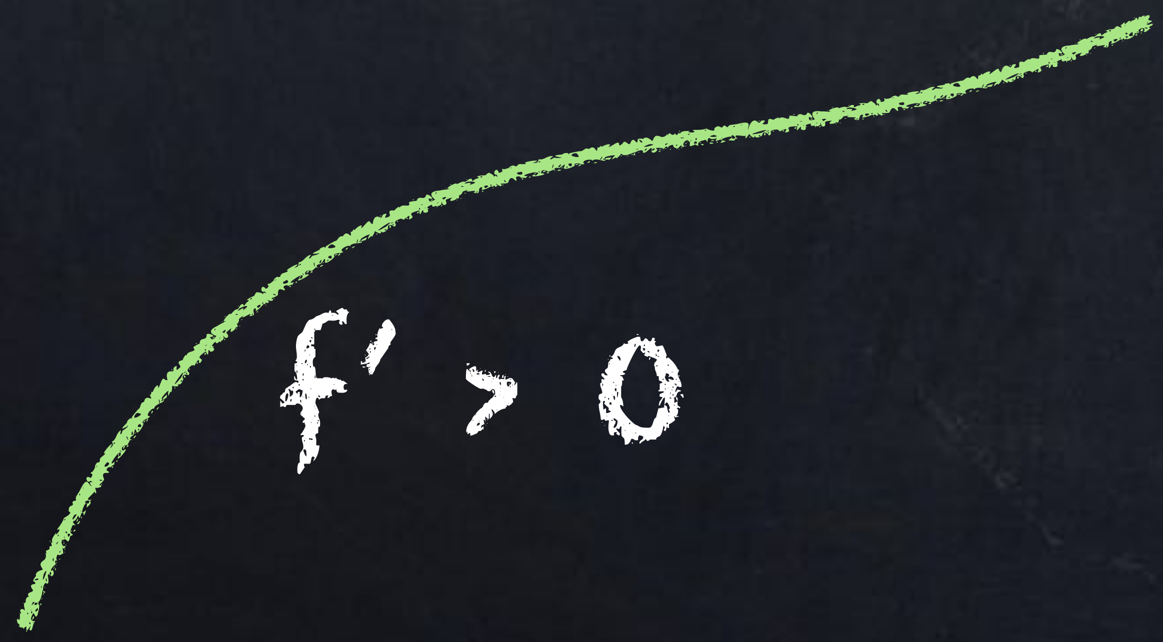
If we know what happens to f(x) when x increases, we know the opposite happens when x decreases.

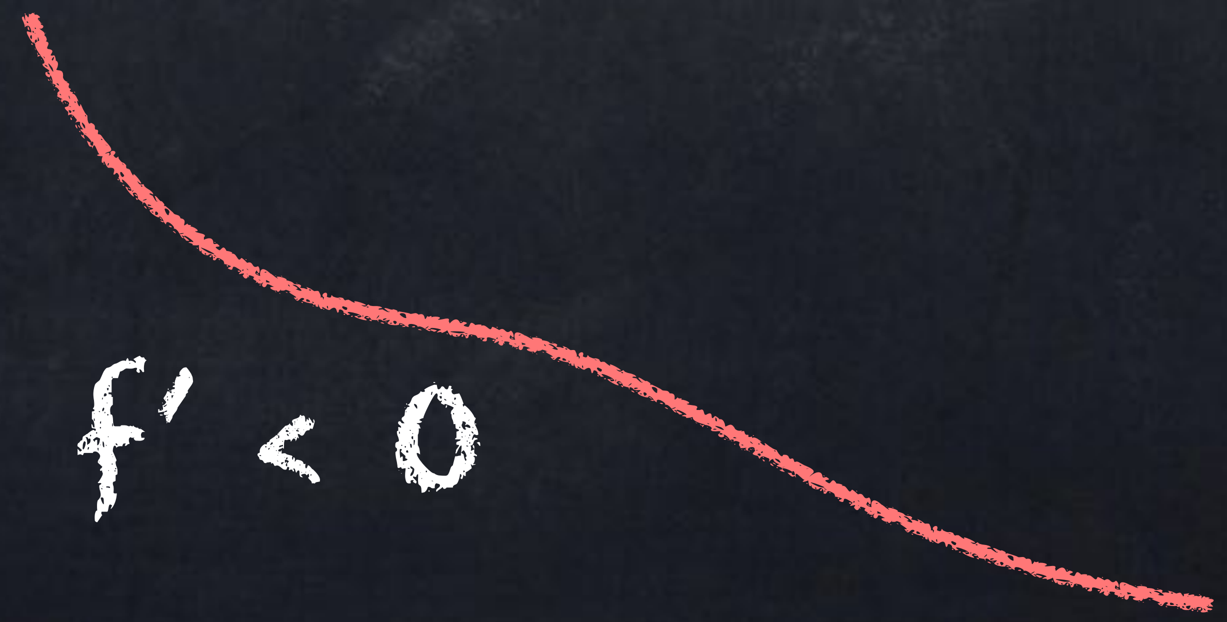
Increasing and decreasing

Definitions: We say $f(x)$ is <u>strictly **increasing** on an interval</u> if for any a, b in that interval with $a < b$ we have $f(b) > f(a)$.

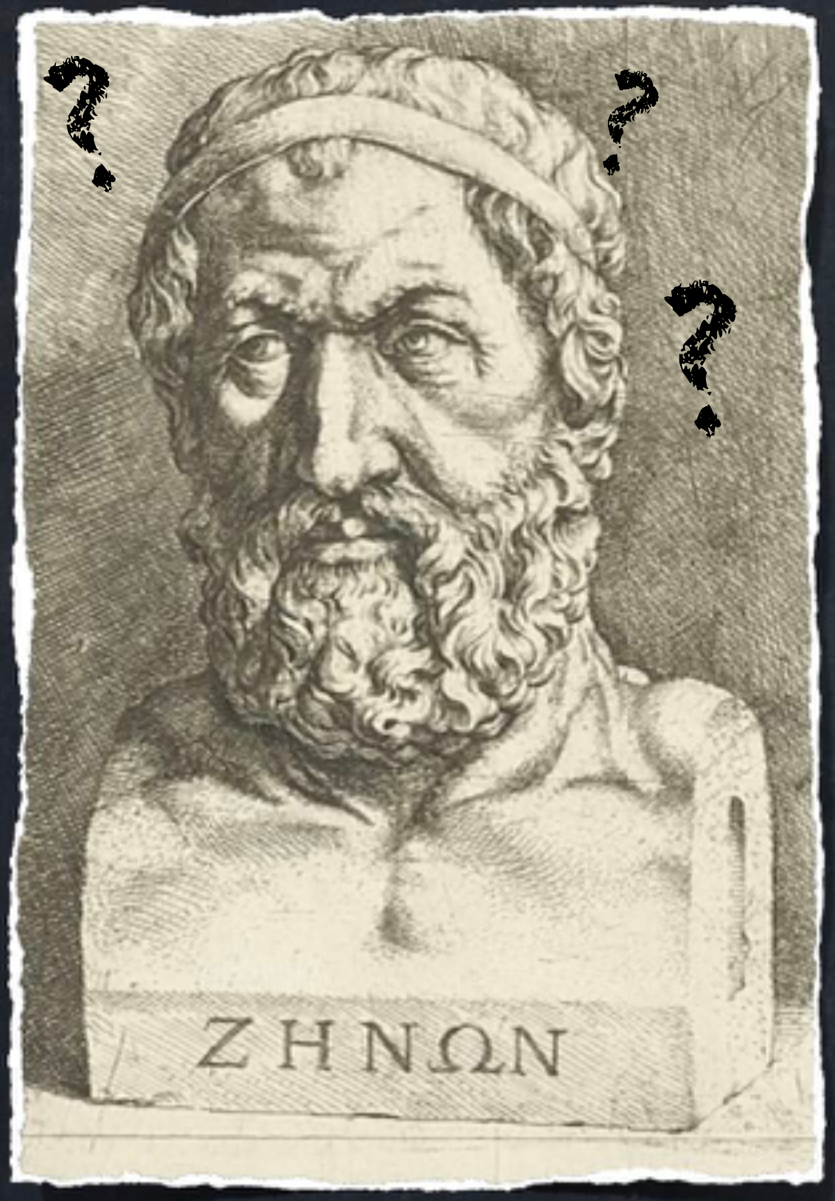
We say $f(x)$ is strictly **decreasing** on an interval if … $f(b) < f(a)$.

Facts: If $f(x)$ is strictly increasing on an interval, then $f'(x) > 0$ for all x-values in that interval. If $f(x)$ is strictly decreasing, then $f'(x) < 0$.





Increasing and decreasing



Zeno of Elea wrote three Paradoxes of Motion, such as asking how an arrow can possibly move if it is motionless at every instant in time.

Today, limits and derivatives can resolve many of his questions.

Definition: We say $f(x)$ is **increasing at $x = c$** if $f'(c) > 0$, and we say $f(x)$ is **decreasing at $x = c$** if $f'(c) < 0$.

- You can argue philosophically about whether f can really be increasing when x is exactly c, but it is a helpful word to use.

Increasing and decreasing

- Intervals where this functions is increasing

  $-1 < x < 2$

  and $4 < x < 5$

- Intervals where this function is decreasing

  $x < -1$

  and $2 < x < 4$

  and $x > 5$

You can write $(-1,2) \cup (4,5)$ and $(-\infty,-1) \cup (2,4) \cup (5,\infty)$ if you prefer.

Critical points

A **critical point** of f is an x-value in the domain of f at which $f'(x)$ is either 0 or doesn't exist.

- f' is zero → graph of f has a horizontal tangent line;
- f' does't exist → graph of f has a vertical tangent line, or corner, or jump.

A function can only change from increasing to decreasing (or dec. to inc.) at a critical point.

Task 1: Find the critical point(s) of $f(x) = \frac{1}{8}x^4 - 4x + 3$.

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x = 2 only

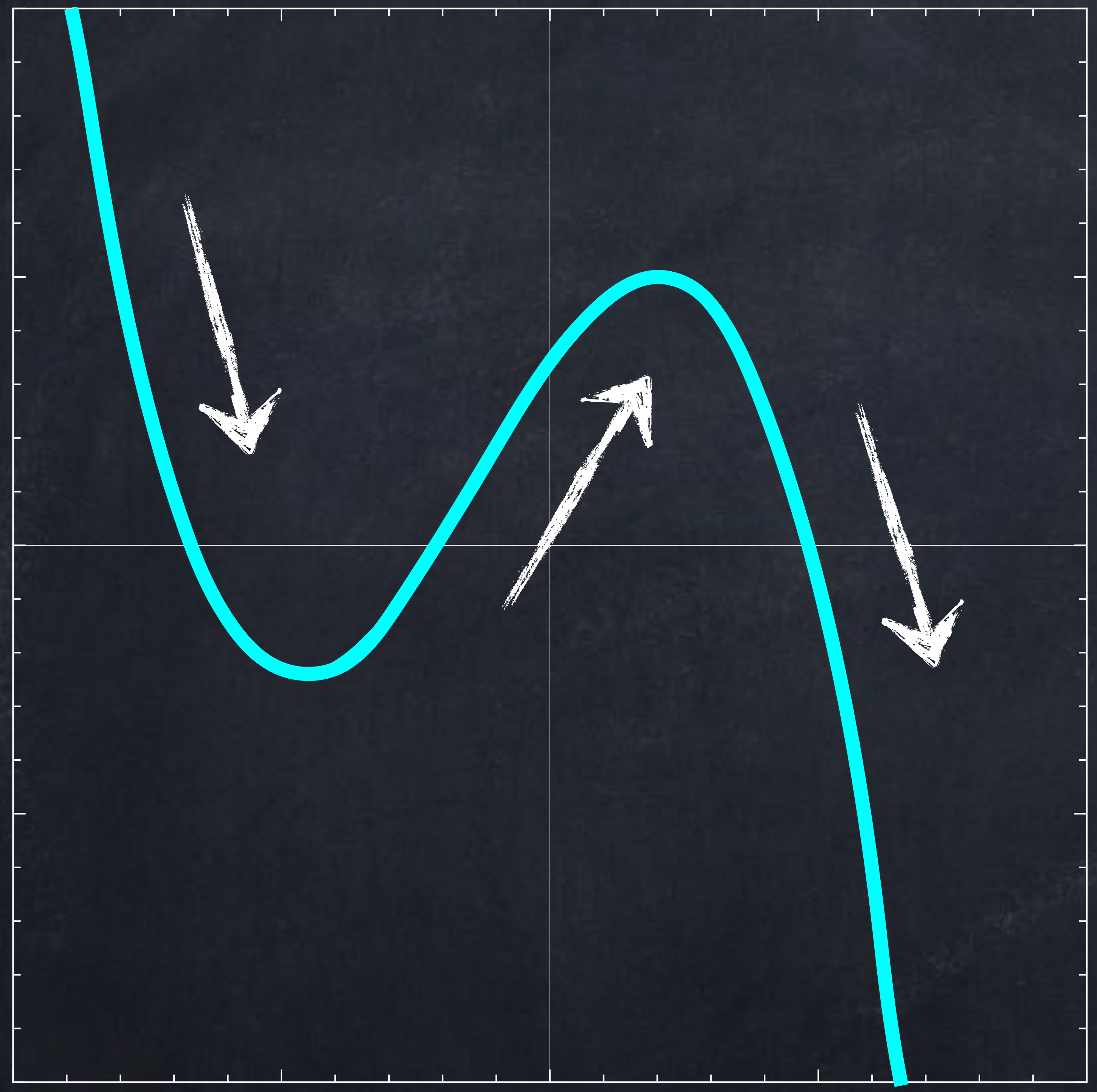
- Task 2: On what interval(s) is $f(x) = \frac{1}{8}x^4 - 4x + 3$ increasing?

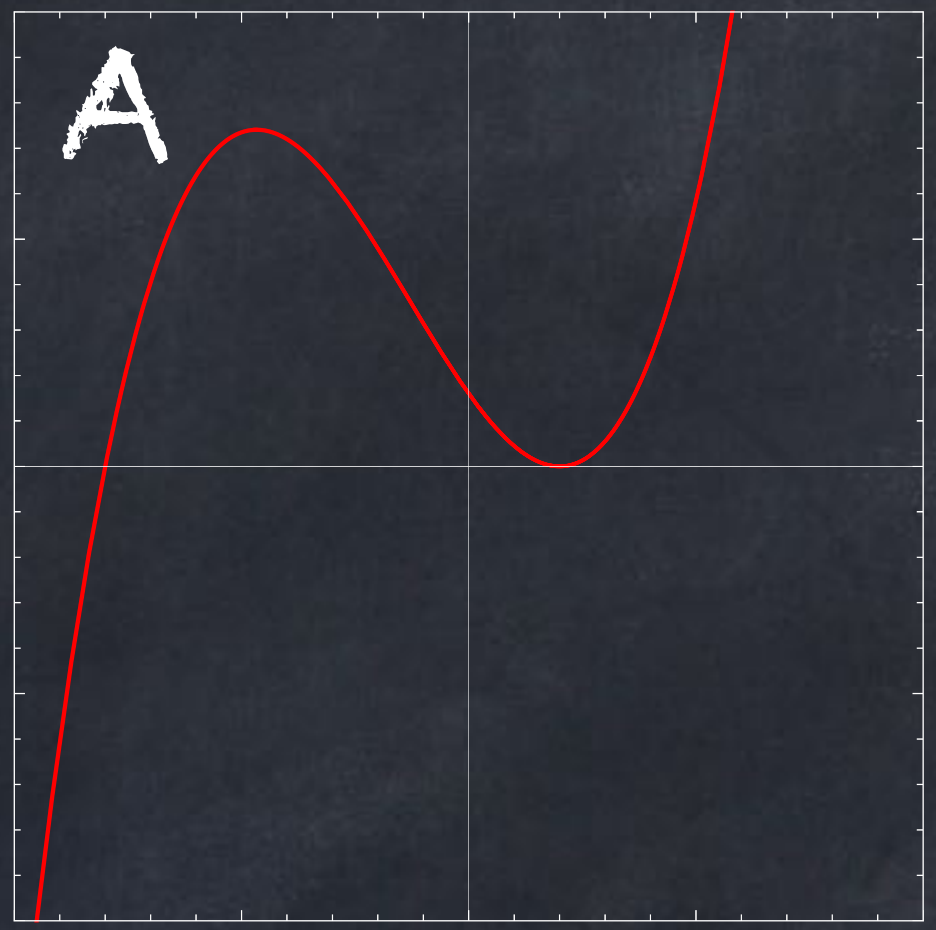
$x > 2$ or $(2, \infty)$

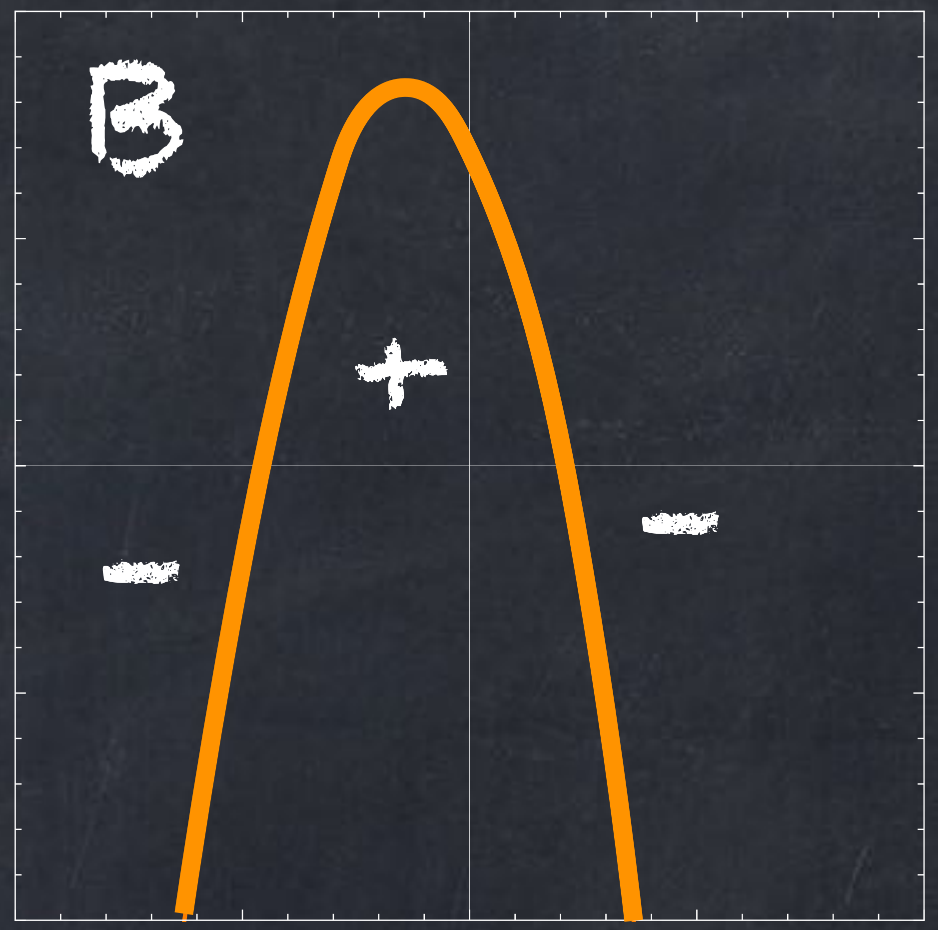
- Task 3: On what interval(s) is $f(x) = \frac{1}{8}x^4 - 4x + 3$ decreasing?

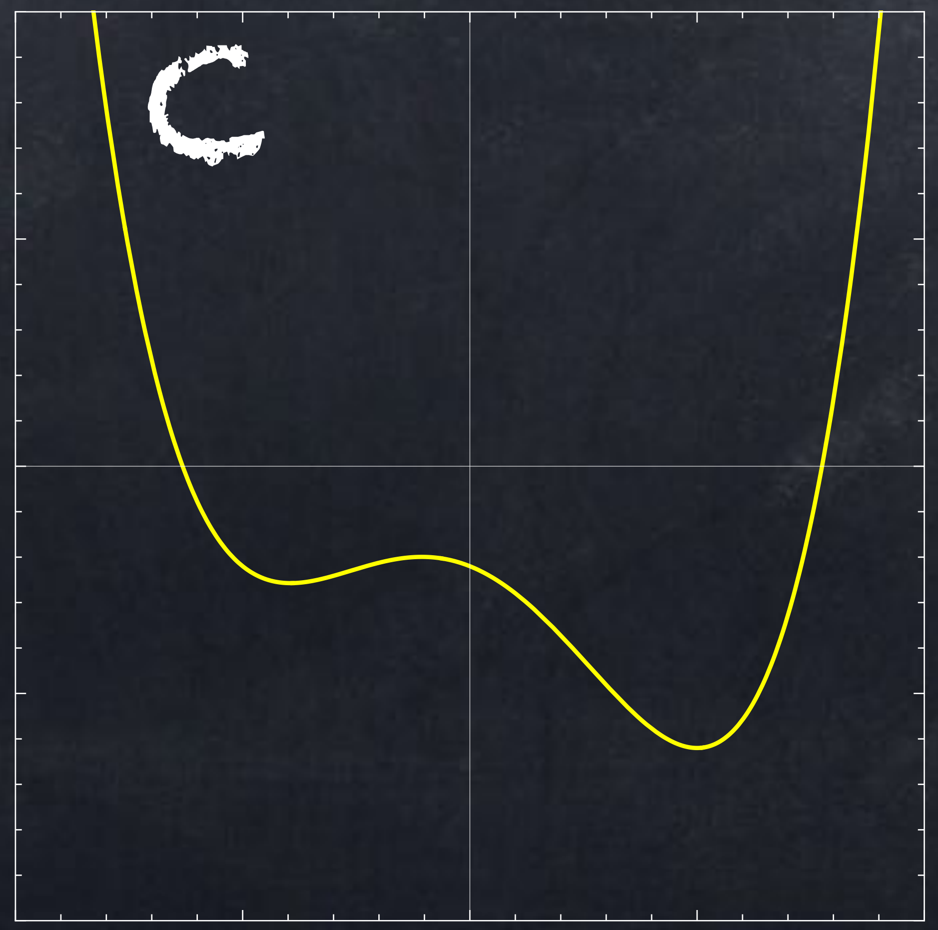
$x < 2$ or $(-\infty, 2)$

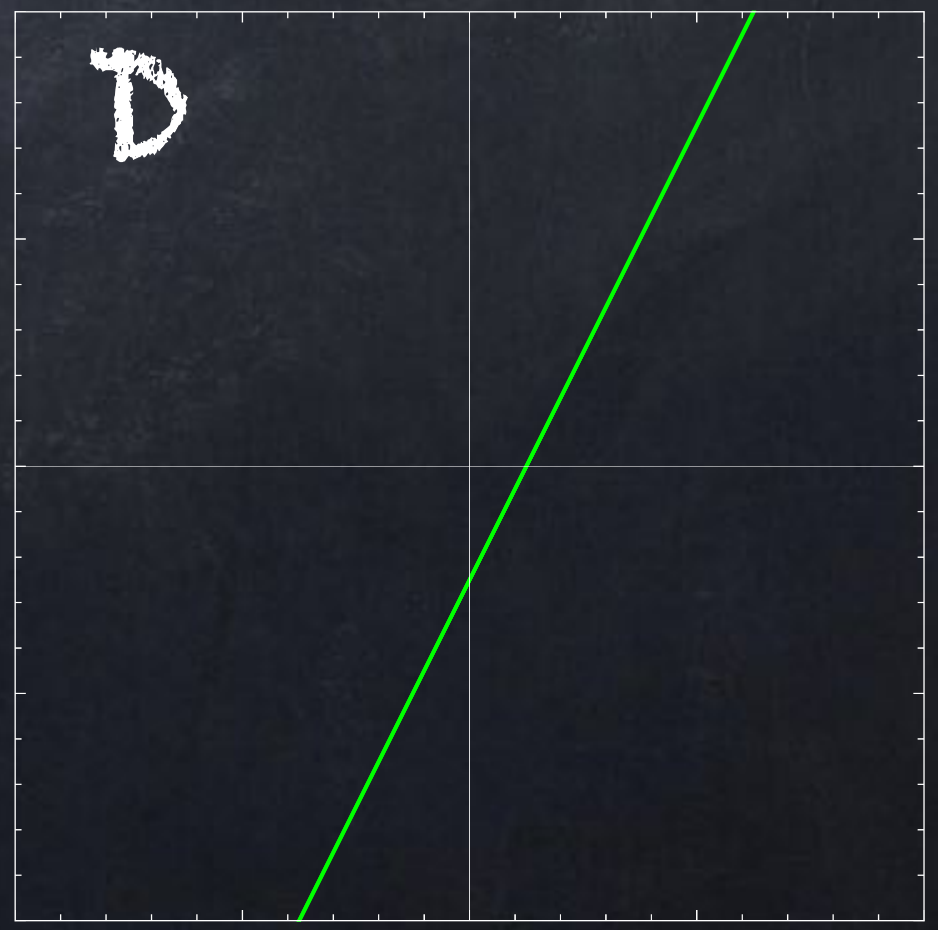
Which of the graphs A-D is the derivative of the graph on the left?











Critical points

A number c is a “critical point” of $f(x)$ …

- if $f'(c) = 0$ (horizontal tangent line) or
- if $f'(c)$ doesn’t exist (vertical tangent line, or corner, or jump).

Increasing and decreasing

On an interval *or* at a single point,

- if $f' > 0$ then f is increasing,
- if $f' < 0$ then f is decreasing.

Minimum and maximum

- How do these relate to derivatives?

Extrema (min and max)

We say $f(x)$ has…

- an **absolute maximum at $x = c$** if $f(c) \geq f(x)$ for all allowed x values.
- an **absolute minimum at $x = c$** if $f(c) \leq f(x)$ for all allowed x values.
- a **local maximum at $x = c$** if $f(c) \geq f(x)$ for all x in some open interval containing c.
- a **local minimum at $x = c$** if $f(c) \leq f(x)$ for all x in some open interval containing c.

- an **absolute extreme** if it has a absolute max *or* absolute min.
- a **local extreme** if it has a local max *or* local min.

Extrema (min and max)

The plural of maximum is *maxima* or *maximums*.
Or you can just write “maxs” or “max”.

1 minimum (or 1 min) → 2 minima or 2 minimums or 2 mins or 2 min.

1 extremum (or 1 extreme) → 2 extrema or 2 extremes.

Types of extremes

- Local maximum(s)

  A C E

- Local minimum(s)

  B D

- Absolute maximum(s)

  C E

- Absolute minimum(s)

  (none)

Types of extremes

What if we only look at the interval $0 \leq x \leq 3$?

- Absolute maximum(s)

  C (x = 1)

- Absolute minimum(s)

  At x = 3, y = −1.